

On spanning trees with high internal degree

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Abstract

Alon and Wormald showed that any graph with minimum degree d contains a spanning star forest in which every connected component is of size at least $\Omega((d/\log d)^{1/3})$. They asked if any connected graph with minimum degree at least d has a spanning tree in which every internal vertex has degree at least $cd/\log d$, for some absolute constant $c > 0$.

We give a simple example showing that this is not the case.

1 The example

Let G be any graph. A *star factor* of G is a spanning forest of G in which every connected component is a star. Answering a question of Havet, Klazar, Kratochvil, Kratsch and Liedloff ([2]), Alon and Wormald ([1]) proved the following result.

Theorem 1 ([1]). *There exists a $c > 0$ such that for all $d \geq 2$, every graph with minimum degree d contains a star factor in which every star has at least $c \left(\frac{d}{\log d}\right)^{1/3}$ edges.*

As noted by Alon and Wormald, any n -vertex connected graph of minimum degree d has a spanning tree with at least $n - O(n \frac{\log d}{d})$ leaves (see [3]). In view of this and Theorem 1, they asked if there exists a constant $c > 0$, such that any connected graph of minimum degree at least d has a spanning tree in which every internal vertex is of degree at least $c \frac{d}{\log d}$.

We now describe an example showing that this need not hold.

Theorem 2. *For any $d \geq 2$ and $n \geq d(d+2)$, there exists a connected graph on n vertices with minimum degree d , for which every spanning tree has an internal vertex of degree 2.*

Proof. Take a complete graph K on d vertices and fix some vertex $x \in K$. For every vertex $y \in K \setminus \{x\}$, add a complete graph K_y on $d+1$ vertices, and join y to some vertex $z_y \in K_y$. Finally, add a complete graph W on $n - (d-1)(d+1) - d$ vertices, and join x to some vertex $z_x \in W$. Let G be the resulting graph.

As $|W| \geq d+1$, any vertex in G has degree at least d , and every vertex in K has degree exactly d .

Let T be any spanning tree of G . Then T must contain all the edges $\{(u, z_u) : u \in K\}$, and furthermore all the vertices in K are internal vertices of T . Moreover, T induces a tree $T[K]$ on K . Let ℓ be any leaf of $T[K]$. Then ℓ is an internal vertex of T of degree 2. This completes the proof. \square

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The graph G in Theorem 2 can also be modified to be regular of degree d (albeit for a larger lower bound on n).

References

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